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P-kernels, IC bases and Kazhdan–Lusztig polynomials

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Abstract

In [J. Amer. Math. Soc. 5 (1992) 805–851] Stanley introduced the concept of a P-kernel for any locally finite partially ordered set P. In [Proc. Sympos. Pure Math., Vol. 56, AMS, 1994, pp. 135–148] Du introduced, for any set P, the concept of an IC basis. The purpose of this article is to show that, under some mild hypotheses, these two concepts are equivalent, and to characterize, for a given Coxeter group W, partially ordered by Bruhat order, the W-kernel corresponding to the Kazhdan–Lusztig basis of the Hecke algebra of W. Finally, we show that this W-kernel factorizes as a product of other W-kernels, and that these provide a solution to the Yang–Baxter equations for W.

1. Introduction

In his 1992 article [18] Stanley introduced, for any locally finite poset P, the concept of a P-kernel. This concept includes as special cases several interesting objects, including Kazhdan–Lusztig polynomials, and the local intersection homology Poincaré polynomials of toric varieties. Furthermore, several results on these polynomials actually generalize to the more general setting of P-kernels (see, e.g., [18, Part II], [5,6]). In an independent development, Du in 1994 introduced in [10], for any set P, the concept of an IC basis. IC bases include as special cases many interesting bases such as Kazhdan–Lusztig bases of Hecke algebras of Coxeter groups and of q-Schur algebras, as well as canonical bases of

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quantized enveloping algebras, of quantum linear groups, and of generalized Temperley–Lieb algebras.

The purpose of this article is to show that, under some mild hypotheses, P-kernels and IC bases are equivalent, and to characterize, given a Coxeter group W partially ordered by Bruhat order, among all W-kernels the one that corresponds to the Kazhdan–Lusztig basis of the Hecke algebra of W (we call this the *Kazhdan–Lusztig kernel* of W). We also show that this W-kernel factorizes as a product of other W-kernels and that these provide a solution to the Yang–Baxter equations for W, and then we compute explicitly the IC bases corresponding to them.

The organization of the article is as follows. In the next section we collect some definitions and results that we need in the sequel. In Section 3 we show the equivalence, given a locally finite poset P, of P-kernels and IC bases. In Section 4 we give a characterization, among all possible W-kernels, of the Kazhdan–Lusztig kernel of a Coxeter group W. In Section 5 we show that the Kazhdan–Lusztig kernel factorizes as a product of other W-kernels, we compute explicitly the IC bases corresponding to these factors, and show that they provide a solution to the Yang–Baxter equations for W, thereby generalizing to any Coxeter group a special case of the main result of [7].

2. Definitions, notation, and preliminaries

In this section we collect some definitions, notation and results that will be used in the rest of this work. We let $\mathbf{P} \stackrel{\text{def}}{=} \{1, 2, 3, \ldots\}$, $\mathbf{N} \stackrel{\text{def}}{=} P \cup \{0\}$, $\mathbf{Z}$ be the ring of integers, $\mathbf{Q}$ be the field of rational numbers, and $\mathbf{R}$ be the field of real numbers. By a *directed graph* we mean a pair $D = (V, E)$ where V is a set and $E \subseteq V^2$. We call the elements of V *vertices* and those of E *directed edges.* A *directed path* in D is a sequence $\Gamma = (a_0, \ldots, a_r)$ of vertices such that $(a_{i-1}, a_i) \in E$ for $i = 1, \ldots, r$, we then say that Γ goes from a_0 to a_r. The *length* of such a directed path Γ is $l(\Gamma) \stackrel{\text{def}}{=} r$. If $A \subseteq V$ then the directed graph *induced* on A by D is $(A, E \cap A^2)$.

We follow [17, Chapter 3], for notation and terminology concerning partially ordered sets. In particular, given a partially ordered set (or, poset, for short) P we let $\mathrm{Int}(P) \stackrel{\text{def}}{=} \{(u, v) \in P^2\colon u \leqslant v\}$, and given $u, v \in P$ we let $[u, v] \stackrel{\text{def}}{=} \{x \in P\colon u \leqslant x \leqslant v\}$, and call this an *interval* of P. We consider $[u, v]$ as a poset with the partial ordering induced by P. We say that a poset P is *locally finite* if $|[u, v]| < +\infty$ for all $(u, v) \in \mathrm{Int}(P)$, and we denote by δ_P the *delta* function of P. We will usually omit the index P if there is no danger of confusion.

Recall (see, e.g., [17, Section 3.6]) that, given a locally finite poset P and a commutative ring R, the *incidence algebra* of P with coefficients in R, denoted $I(P; R)$, is the set of all functions $f : \mathrm{Int}(P) \to R$ with sum and product defined by

$$(f + g)(u, v) \stackrel{\text{def}}{=} f(u, v) + g(u, v) \quad \text{and} \quad (fg)(u, v) \stackrel{\text{def}}{=} \sum_{u \leqslant z \leqslant v} f(u, z)g(z, v)$$

for all $f, g \in I(P; R)$ and $(u, v) \in \mathrm{Int}(P)$. It is well known (see, e.g., [17, Section 3.6 and Proposition 3.6.2]) that $I(P; R)$ is an associative algebra having δ as identity element, and that an element $f \in I(P;R)$ is invertible if and only if $f(u,u)$ is invertible in R for all $u \in P$. If f is invertible then we denote by f^{-1} its (two-sided) inverse. We adopt the convention that $f(u,v) \stackrel{\text{def}}{=} 0$ if $f \in I(P;R)$ and $u, v \in P$, $u \nleqslant v$.

Let P be a locally finite poset, and $r : P \to \mathbf{Z}$ be such that if $u < v$ then $r(u) < r(v)$. Then $\rho(u,v) \stackrel{\text{def}}{=} r(v) - r(u)$ is a weak rank function for P in the sense of [6]. Let $I(P) \stackrel{\text{def}}{=} I(P;\mathbf{R}[q])$. Following Stanley (see [18, p. 830 and Proposition 6.11, p. 835]) we let

$$\tilde{I}(P) \stackrel{\text{def}}{=} \bigl\{ f \in I(P)\colon \deg\bigl(f(u,v)\bigr) \leqslant \rho(u,v), \text{ for all } (u,v) \in \operatorname{Int}(P) \bigr\},$$

and

$$I_{1/2}(P) \stackrel{\text{def}}{=} \bigl\{ f \in \tilde{I}(P)\colon \deg\bigl(f(u,v)\bigr) < \tfrac{1}{2}\rho(u,v), \text{ for all } u < v, \text{ and } f(u,u) = 1 \bigr\}.$$

Note that $\tilde{I}(P)$ is a subalgebra of $I(P)$ and that, if $f \in I(P)$ is invertible, then $f \in \tilde{I}(P)$ if and only if $f^{-1} \in \tilde{I}(P)$. Given $f \in \tilde{I}(P)$ we let

$$\bar{f}(u,v) \stackrel{\text{def}}{=} q^{\rho(u,v)} f(u,v)\left(\frac{1}{q}\right),$$

for all $(u,v) \in \operatorname{Int}(P)$. Notice that $\tilde{I}(P)$, $I_{1/2}(P)$, and the involution $^{-}$ are dependent on r. However, throughout this work r is fixed, so no confusion should arise.

Following [18, Definition 6.2, p. 830] we say that an element $K \in I(P)$ is a P-*kernel* (or, more simply, a *kernel*) if K is *unitary* (i.e., $K(u,u) = 1$ for all $u \in P$) and there exists an element $f \in I(P)$ such that:

(i) f is invertible in $I(P)$;
(ii) $Kf = \bar{f}$.

An element $f \in I(P)$ satisfying (ii) above is called K-*totally acceptable* (see [18, Definition 6.2, p. 830]).[2] The next result was first proved by Stanley in the locally graded case (see [18, Corollary 6.7]), and by the author in the locally finite one (see [6, Theorem 6.2]).

Theorem 2.1. *Let P be a locally finite poset and $K \in I(P)$ a P-kernel. Then there exists a unique K-totally acceptable element $\gamma \in I_{1/2}(P)$.*

We call the element γ whose existence and uniqueness is guaranteed by the preceding theorem the *Kazhdan–Lusztig–Stanley function* (or *KLS-function*, for short) of K. As noted in [18, Sections 6 and 7], the function γ specializes to many interesting objects depending on the particular choice of the poset P and kernel K.

[2] These definitions are slightly different from those in [18]: there they are "left-handed" (Stanley uses "fK" instead of "Kf". This choice is more convenient for our purposes, and does not affect the validity of any of the results quoted in this section and used in the sequel.

There is a simple way to decide if a given element $K \in I(P)$ is a P-kernel or not. The following result was first proved by Stanley in [18, Theorem 6.5, p. 831] in the case that P is locally graded. However, his proof carries over unchanged to the present, more general setting.

Theorem 2.2. *Let P be a locally finite poset and $K \in I(P)$ be unitary. Then K is a P-kernel if and only if $K\overline{K} = \delta$.*

Note that Theorem 2.1 defines a map from the set of P-kernels to $I_{1/2}(P)$ and that, by Theorem 2.2, the map $f \mapsto \bar{f} f^{-1}$ is its inverse. Thus the correspondence $K \mapsto \gamma$ in Theorem 2.1 is a bijection. We call this bijection the *KLS-correspondence* of P and the elements of $I_{1/2}(P)$ the *KLS-functions* of P.

Let P be a countable set, and M be the free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-module with basis $\{m_i\}_{i \in P}$. Let $j : M \to M$ be a $\mathbf{Z}$-linear involution such that $j(q) = q^{-1}$ and $j(am) = j(a)j(m)$ for all $m \in M$, $a \in \mathbf{Z}[q^{1/2}, q^{-1/2}]$, and let $r : P \to \mathbf{Z}$. Let $\mathcal{L}$ be the free $\mathbf{Z}[q^{-1/2}]$-module with basis $m'_k \stackrel{\text{def}}{=} q^{-r(k)/2} m_k$ $(k \in P)$. So

$$\mathcal{L} = \left\{ \sum_{i \in P} a_i\left(q^{-1/2}\right) q^{-r(i)/2} m_i \colon\ a_i(t) \in \mathbf{Z}[t] \right\}.$$

Clearly, $\mathcal{L}$ is a $Z[q^{-1/2}]$-submodule of M (seen as a $\mathbf{Z}[q^{-1/2}]$-module).

Following [10] we say that a basis $\{c_i\}_{i \in P}$ of $\mathcal{L}$ is the *IC basis* of M with respect to $(\{m_i\}_{i \in P}, j, \mathcal{L})$ if, for all $i \in P$, we have that:

(i) $j(c_i) = c_i$;
(ii) $\pi(c_i) = \pi(m'_i)$, where $\pi : \mathcal{L} \to \mathcal{L}/q^{-1/2}\mathcal{L}$ is the canonical projection;
(iii) $\{c_i\}_{i \in P}$ is the only basis of $\mathcal{L}$ satisfying (i) and (ii).

Examples of IC bases include Kazhdan–Lusztig bases of Hecke algebras of Coxeter groups and of q-Schur algebras, as well as canonical bases of quantized enveloping algebras, quantum linear groups, and generalized Temperley–Lieb algebras (see [10,14] for details and further references).

We follow notation and terminology of [15] for general Coxeter groups. Given a Coxeter system (W, S) and $v \in W$, we denote by $l(v)$ the length of v in W with respect to S, and we let

$$D(v) \stackrel{\text{def}}{=} \left\{ s \in S \colon\ l(sv) < l(v) \right\} \quad \text{and} \quad \varepsilon_v \stackrel{\text{def}}{=} (-1)^{l(v)}.$$

We denote by e the identity of W and let $T \stackrel{\text{def}}{=} \{vsv^{-1}\colon\ v \in W,\ s \in S\}$ be the set of *reflections* of W. For $u, v \in W$ we also write $l(u, v) \stackrel{\text{def}}{=} l(v) - l(u)$. We denote by $B(W)$ the *Bruhat graph* of W. Recall (see, e.g., [15, Section 8.6] or [12]) that this is the directed graph with W as a vertex set and with a directed edge from u to v if and only if $u^{-1}v \in T$ and $l(u) < l(v)$ (we then write $u \to v$). If $u^{-1}v = r \in T$, we also write $u \xrightarrow{r} v$. The transitive closure of $B(W)$ is a partial order on W, usually called the *Bruhat order* (see, e.g., [15, Section 5.9]); we denote it by $\leqslant$. Throughout this work we always assume that W is partially ordered by $\leqslant$.

Let $A \subseteq T$ and W' be the subgroup of W generated by A. Following [15, Section 8.2], we call W' a *reflection subgroup* of W. It is known (see, e.g., [15, Theorem 8.2]) that there is another Coxeter system (W', S') where $S' \stackrel{\text{def}}{=} \{t \in T\colon\ N(t) \cap W' = \{t\}\}$ and $N(w) \stackrel{\text{def}}{=} \{t \in T\colon\ l(wt) < l(w)\}$, and that $T \cap W' = \{vtv^{-1}\colon\ t \in S',\ v \in W'\}$ [11, Theorem 3.3(i)]. We call the elements of S' the *canonical generators* of W'. We say that W' is a *dihedral reflection subgroup* if $|S'| = 2$ (i.e., if (W', S') is a dihedral Coxeter system). Following [13], we say that a total ordering $\prec$ of T is a *reflection ordering* if, for any dihedral reflection subgroup W' of W, either

$$a \prec aba \prec ababa \prec \cdots \prec babab \prec bab \prec b \quad \text{or}$$
$$b \prec bab \prec babab \prec \cdots \prec ababa \prec aba \prec a$$

where $\{a, b\} \stackrel{\text{def}}{=} S'$. The existence of reflection orderings (and many of their properties) is proved in [13, Section 2] and [1, Section 5.2].

A family $\{R_\tau\}_{\tau \in T}$ of elements of a monoid is called an (extensible) *solution to the Yang–Baxter equations* for W if, for any dihedral reflection subgroup W' of W with canonical generators a and b, it yields

$$R_a R_{aba} R_{ababa} \cdots R_{bab} R_b = R_b R_{bab} \cdots R_{ababa} R_{aba} R_a. \tag{1}$$

The collection $\{R_\tau\}_{\tau \in T}$ satisfying the Yang–Baxter equations (1) is sometimes called an (extensible) R-matrix (of the corresponding type). In the case of a Weyl group, Eqs. (1), stated case by case in terms of the root system, were given by I.V. Cherednik (implicitly in [8] and explicitly in [9, Definition 2.1a]).

Let W' be a reflection subgroup of W and S' be its set of canonical generators. Let $T' \stackrel{\text{def}}{=} T \cap W'$ and l' be the length function of W' with respect to S'. Given $u \in W$, it is known (see [11, Corollary 3.4(ii)]) that there is a unique element w_0 of uW' having minimal length. We then denote u_0 the unique element of W' such that $u = w_0 u_0$. The following result is an immediate consequence of [11, Theorem 3.3(i) and Corollary 3.4].

Proposition 2.3. *Let W' be a reflection subgroup of W and $u, v \in W$ be such that $v^{-1}u \in W'$. Then, for each $r \in T \cap W'$,*

$$u \xrightarrow{r} v \text{ in } B(W) \quad \Leftrightarrow \quad u_0 \xrightarrow{r} v_0 \text{ in } B(W) \quad \Leftrightarrow \quad u_0 \xrightarrow{r} v_0 \text{ in } B(W').$$

We denote by $\mathcal{H}(W)$ the *Hecke algebra* associated to W. Recall (see, e.g., [15, Chapter 7]) that this is the free $\mathbf{Z}[q, q^{-1}]$-module having the set $\{T_w\colon\ w \in W\}$ as a basis with multiplication defined so that

$$T_s T_w = \begin{cases} T_{sw}, & \text{if } l(sw) > l(w), \\ qT_{sw} + (q-1)T_w, & \text{if } l(sw) < l(w), \end{cases} \tag{2}$$

for all $w \in W$ and $s \in S$. It is well known that this is an associative algebra having T_e as identity and that each basis element is invertible in $\mathcal{H}(W)$. More precisely, we have the following result (see [15, Proposition 7.4]).

Proposition 2.4. *Let $v \in W$. Then*

$$(T_{v^{-1}})^{-1} = q^{-l(v)} \sum_{u \leqslant v} (-1)^{l(v)-l(u)} R_{u,v}(q) T_u, \quad \textit{where } R_{u,v}(q) \in \mathbf{Z}[q].$$

The polynomials $R_{u,v}$ defined by the previous proposition are called the *R-polynomials* of W. It is easy to see that $\deg(R_{u,v}) = l(v) - l(u)$, and that $R_{u,u} = 1$ for all $u, v \in W$, $u \leqslant v$. It is customary to let $R_{u,v} \stackrel{\text{def}}{=} 0$ for $u \nleqslant v$. We then have the following fundamental result that follows from (2) and Proposition 2.4 (see [15, Section 7.5]).

Theorem 2.5. *Let $u, v \in W$ and $s \in D(v)$. Then*

$$R_{u,v}(q) = \begin{cases} R_{su,sv}(q), & \textit{if } s \in D(u), \\ q R_{su,sv}(q) + (q-1) R_{u,sv}(q), & \textit{if } s \notin D(u). \end{cases}$$

Define an element $\mathcal{R} \in I(W)$ by letting

$$\mathcal{R}(u, v) \stackrel{\text{def}}{=} R_{u,v}(q) \tag{3}$$

for all $(u, v) \in \operatorname{Int}(W)$. The next result is well known (although it is not usually stated in this form); a proof of it can be found, e.g., in [15, Proposition 7.8] (see also [18, Example 6.9]).

Proposition 2.6. *$\mathcal{R}$ is a W-kernel.*

We call $\mathcal{R}$ the *Kazhdan–Lusztig kernel* of W. Let $\mathcal{P}$ be the KLS-function of $\mathcal{R}$. It then follows immediately from the definitions and well-known results (see, e.g., [15, Section 7.10]) that

$$\mathcal{P}(u, v) = P_{u,v}(q)$$

for all $(u, v) \in \operatorname{Int}(W)$, where $\{P_{u,v}(q)\}_{u,v \in W}$ are the *Kazhdan–Lusztig polynomials* of W.

3. P-kernels and IC bases

In this section we show the equivalence, given a locally finite poset P, of P-kernels and IC bases.

Let $(P, \leqslant)$ be a locally finite poset, $r : P \to \mathbf{Z}$, and M be the free $\mathbf{Z}[q^{1/2}, q^{-1/2}]$-module with basis $\{m_i\}_{i \in P}$.

Proposition 3.1. *Let $K \in I(P)$ be unitary. Then the following are equivalent*:

(i) *K is a P-kernel*;

(ii) *the map* $\iota : M \to M$ *defined by*

$$\iota(m_j) \stackrel{\text{def}}{=} q^{-r(j)} \sum_{i \in P} K(i, j) m_i,$$

for $j \in P$, $\iota(q) = q^{-1}$, *and* **Z**-*linear extension, is an involution of* M.

Proof. We have that

$$\begin{aligned}\iota^2(m_j) &= \sum_{i \in P} q^{r(j)} K(i, j) \left(\frac{1}{q}\right) \iota(m_i) = \sum_{i \leqslant j} q^{r(j)} \overline{K}(i, j) \sum_{k \in P} q^{-r(i)} K(k, i) m_k \\ &= \sum_{k \in P} \left(\sum_{k \leqslant i \leqslant j} K(k, i) \overline{K}(i, j) \right) m_k.\end{aligned}$$

Therefore $\iota^2(m_j) = m_j$ for all $j \in P$ if and only if

$$\sum_{k \leqslant i \leqslant j} K(k, i) \overline{K}(i, j) = \delta(k, j)$$

for all $k, j \in P$, and the result follows from Theorem 2.2. □

Keeping the notation as in the previous proposition, we can now prove the main result of this section. Note that by Theorem 2.2, $K \in I(P)$ is a P-kernel if and only if $\overline{K}$ is.

Theorem 3.2. *Let* $K \in I(P)$ *be a* P-*kernel, and* $\gamma \in I(P)$ *be unitary. Then the following are equivalent*:

(i) γ *is the KLS-function of* $\overline{K}$;
(ii) $\{q^{-r(i)/2} \sum_{k \in P} \gamma(k, i) m_k\}_{i \in P}$ *is the IC basis of* M *with respect to* $(\{m_i\}_{i \in P}, \iota, \mathcal{L})$.

Proof. Let, for brevity, $c_i \stackrel{\text{def}}{=} q^{-r(i)/2} \sum_{k \in P} \gamma(k, i) m_k$ for $i \in P$. We then have that

$$\begin{aligned}\iota(c_i) &= \sum_{k \in P} q^{r(i)/2} \gamma(k, i) \left(\frac{1}{q}\right) \iota(m_k) = \sum_{k \in P} q^{-r(i)/2} \gamma(k, i) \sum_{j \in P} K(j, k) m_j \\ &= \sum_{j \in P} \left(\sum_{j \leqslant k \leqslant i} K(j, k) \bar{\gamma}(k, i) \right) q^{-r(i)/2} m_j\end{aligned}$$

for all $i \in P$. Therefore $\iota(c_i) = c_i$ for all $i \in P$ if and only if

$$\sum_{j \leqslant k \leqslant i} K(j, k) \bar{\gamma}(k, i) = \gamma(j, i)$$

for all $i, j \in P$, namely if and only if $\overline{K} \gamma = \bar{\gamma}$ in $I(P)$.

Furthermore, $c_i \in \mathcal{L}$ for all $i \in P$ if and only if $\gamma(k,i)q^{(r(k)-r(i))/2} \in \mathbf{Z}[q^{-1/2}]$ for all $k, i \in P$, namely if and only if $\deg(\gamma(k,i)) \leqslant (r(i)-r(k))/2$ for all $(k,i) \in \mathrm{Int}(P)$. Finally, $\pi(c_i) = \pi(m_i')$ if and only if $\gamma(k,i)q^{-r(i)/2}m_k \in q^{-1/2}\mathcal{L}$ for all $k \neq i$, which happens if and only if $\deg(\gamma(k,i)) < (r(i)-r(k))/2$ for all $k < i$.

The facts that $\gamma \in I(P)$ and γ is unitary make it clear that $\{c_i\}_{i\in P}$ is a basis of $\mathcal{L}$. Suppose now that $\{d_i\}_{i\in P}$ is another basis of $\mathcal{L}$ such that $\iota(d_i) = d_i$ and $\pi(d_i) = \pi(m_i')$ for all $i \in P$. Then $\iota(d_i - c_i) = d_i - c_i$ and $d_i - c_i \in q^{-1/2}\mathcal{L}$ for all $i \in P$, and hence, by [10, the lemma on p. 138], $d_i - c_i = 0$ for all $i \in P$. $\square$

4. The Kazhdan–Lusztig kernel

In this section we prove the main result of this article. Namely, given a Coxeter group W partially ordered by Bruhat order, we characterize, among all W-kernels, the one that corresponds by Theorems 2.1 and 3.2 to the Kazhdan–Lusztig basis of the Hecke algebra of W.

Let W be a Coxeter group, partially ordered by Bruhat order, and $\mathcal{H}(W)$ be the Hecke algebra of W. Given $K \in I(W)$, we define a map $j : \mathcal{H}(W) \to \mathcal{H}(W)$ by letting

$$j(T_w) \stackrel{\text{def}}{=} q^{-l(w)} \sum_{x \leqslant w} \varepsilon_x \varepsilon_w K(x,w) T_x, \tag{4}$$

for all $w \in W$, $j(q) \stackrel{\text{def}}{=} q^{-1}$, and $\mathbf{Z}$-linear extension. It follows easily from Proposition 3.1 and Theorem 2.2 that K is a W-kernel if and only if j is an involution. We wish to characterize, among all possible W-kernels, the Kazhdan–Lusztig kernel $\mathcal{R}$ defined by (3). The answer turns out to be extremely simple and elegant.

Theorem 4.1. *Let $K \in I(W)$ be a W-kernel. Then the following are equivalent*:

(i) $K = \mathcal{R}$;
(ii) *j is a ring homomorphism.*

Proof. It is well known that if $K = \mathcal{R}$ then the map j defined by (4) is a ring homomorphism (see [15, Section 7.7] and Proposition 2.4).

So let K be a W-kernel, and suppose that j is a ring homomorphism. It is clear that $j(T_e) = T_e$. Furthermore, we have that

$$\begin{aligned} j(T_s^2) &= j\big((q-1)T_s + qT_e\big) = \big(q^{-1}-1\big)j(T_s) + q^{-1}T_e \\ &= \big(q^{-1}-1\big)q^{-1}\big(T_s - K(e,s)T_e\big) + q^{-1}T_e \\ &= \big(q^{-2}-q^{-1}\big)T_s + \big(\big(q^{-1}-q^{-2}\big)K(e,s) + q^{-1}\big)T_e. \end{aligned}$$

On the other hand,

$$\begin{aligned} j(T_s)j(T_s) &= q^{-1}\big(T_s - K(e,s)T_e\big)q^{-1}\big(T_s - K(e,s)T_e\big) \\ &= q^{-2}\big(T_s^2 - 2K(e,s)T_s + K(e,s)^2T_e\big) \end{aligned}$$

$$= q^{-2}\big((q-1)T_s + qT_e - 2K(e,s)T_s + K(e,s)^2 T_e\big)$$
$$= q^{-2}\big(\big(q-1-2K(e,s)\big)T_s + \big(q+K(e,s)^2\big)T_e\big).$$

Hence $q^{-2}(1-q) = q^{-2}(q-1-2K(e,s))$ and $q^{-2}((q-1)K(e,s)+q) = q^{-2}(q + K(e,s)^2)$, so

$$K(e,s) = q-1. \tag{5}$$

Let now $w \in W$ and $s \in S \setminus D(w)$. Then we have that

$$j(T_s T_w) = j(T_{sw}) = -q^{-l(sw)} \sum_{x \in W} \varepsilon_x \varepsilon_w K(x, sw) T_x. \tag{6}$$

On the other hand,

$$\begin{aligned}
j(T_s)j(T_w) &= q^{-1}\big(T_s - K(e,s)T_e\big)q^{-l(w)} \sum_{x \leqslant w} \varepsilon_x \varepsilon_w K(x,w) T_x \\
&= q^{-l(w)-1}\big(T_s + (1-q)T_e\big) \sum_{x \leqslant w} \varepsilon_x \varepsilon_w K(x,w) T_x \\
&= q^{-l(sw)} \varepsilon_w \Bigg[\sum_{x \leqslant w} \varepsilon_x K(x,w)\big(T_s T_x + (1-q)T_x\big)\Bigg] \\
&= q^{-l(sw)} \varepsilon_w \Bigg[\sum_{\{x \leqslant w:\ s \in D(x)\}} \varepsilon_x K(x,w) q T_{sx} \\
&\qquad + \sum_{\{x \leqslant w:\ s \notin D(x)\}} \varepsilon_x \big(K(x,w)T_{sx} + (1-q)K(x,w)T_x\big)\Bigg] \\
&= q^{-l(sw)} \varepsilon_w \Bigg[\sum_{\{x \in W:\ s \in D(x)\}} \varepsilon_x q K(x,w) T_{sx} + \sum_{\{x \in W:\ s \notin D(x)\}} \varepsilon_x K(x,w) T_{sx} \\
&\qquad + \sum_{\{x \in W:\ s \notin D(x)\}} \varepsilon_x (1-q) K(x,w) T_x\Bigg].
\end{aligned} \tag{7}$$

Extracting the coefficient of T_y in (6) and (7) yields that

$$K(y, sw) = \begin{cases} K(sy, w), & \text{if } s \in D(y), \\ qK(sy,w) + (q-1)K(y,w), & \text{if } s \notin D(y), \end{cases}$$

for all $y \in W$. This by (5) and Theorem 2.5 implies, by induction on $l(v)$, that

$$K(y,v) = R_{y,v}(q)$$

for all $y, v \in W$, as desired. $\square$

The KLS-function of the W-kernel $\mathcal{R}$, namely the Kazhdan–Lusztig polynomials of W, have many deep and interesting properties (see, e.g., [15, Chapter 7], or [1, Chapter 5]), including (for finite Coxeter groups, and conjecturally for all Coxeter groups) remarkable nonnegativity and monotonicity properties (see, e.g., either of [2,4,16] for a survey and further references). But the KLS-function of $\mathcal{R}$, is, by Theorem 2.1, uniquely determined by (and uniquely determines) $\mathcal{R}$. Therefore, as surprising as it may seem, Theorem 4.1 implies that all these properties are a consequence of the simple fact that the map j defined by (4) is a ring homomorphism.

5. Factorization and the Yang–Baxter equations

In this section we define, for each reflection $t \in T$, a W-kernel K_t, we compute explicitly the IC bases corresponding to these kernels, and show that they give a solution to the Yang–Baxter equations for W. These kernels are closely related to the Kazhdan–Lusztig kernel since they provide a factorization of it.

Let W be a Coxeter group, and T be its set of reflections. Given $t \in T$, we define an element $K_t \in I(W)$ by letting

$$K_t(u,v) \stackrel{\text{def}}{=} \begin{cases} q^{l(u,v)/2}\bigl(q^{1/2}-q^{-1/2}\bigr), & \text{if } u \xrightarrow{t} v, \\ 1, & \text{if } u = v, \\ 0, & \text{otherwise,} \end{cases}$$

for all $(u,v) \in \mathrm{Int}(W)$. Given $A \subseteq T$ (A finite or infinite) and a total order $\prec$ on A, we define

$$K_A^{\prec} \stackrel{\text{def}}{=} \prod_{t \in A} K_t, \tag{8}$$

the product being in $I(W)$, and the factors being taken in the order given by $\prec$. Note that $K_A^{\prec}$ is well defined since if $(u,v) \in \mathrm{Int}(W)$, then the interval $[u,v]$ is finite, and hence the directed graph induced by $B(W)$ on $[u,v]$ is also finite. So there is a finite subset $B \subseteq A$ such that $K_A^{\prec}(u,v) = K_B^{\prec}(u,v)$.

For $u, v \in W$, we denote by $B_A^{\prec}(u,v)$ the set of all the directed paths $u = a_0 \xrightarrow{t_0} a_1 \xrightarrow{t_1} \cdots \xrightarrow{t_{r-1}} a_r = v$ in $B(W)$ such that:

(i) $t_i \in A$ for all $i = 0, \ldots, r-1$;
(ii) $t_0 \prec t_1 \prec \cdots \prec t_{r-1}$.

In other words, if we label each directed edge $u \xrightarrow{t} v$ in the Bruhat graph with the reflection t, then $B_A^{\prec}(u,v)$ is the set of all the directed paths from u to v in $B(W)$ with edges whose labels are in A, and are increasing, along the path, with respect to the total order $\prec$.

The verification of the next simple result is left to the reader.

Proposition 5.1. *Let $A \subseteq T$ and $\prec$ be a total order on A. Then*

$$K_A^{\prec}(u,v) = q^{l(u,v)/2} \sum_{\Gamma \in B_A^{\prec}(u,v)} \left(q^{1/2} - q^{-1/2}\right)^{l(\Gamma)} \quad \textit{for all } (u,v) \in \mathrm{Int}(W).$$

In particular, using a well-known result on the Kazhdan–Lusztig R-polynomials (see, e.g., [1, Theorem 5.3.4]), we obtain the following factorization result.

Corollary 5.2. *Let $\prec$ be a reflection ordering on T. Then*

$$\mathcal{R} = K_T^{\prec} \quad \textit{in } I(W).$$

Our aim is to show that the elements K_t are W-kernels, to compute their corresponding IC-bases, and to show that they give a solution to the Yang–Baxter equations for W. All these results will follow from the next one, which is the main result of this section.

Let W' be a reflection subgroup of W, and $T' \stackrel{\text{def}}{=} T \cap W'$. Denote by $\leqslant'$ the Bruhat order of W'. We define an element $\mathcal{P}_{T'} \in I(W)$ by letting

$$\mathcal{P}_{T'}(u,v) = \begin{cases} q^{(l(u,v)-l'(u_0,v_0))/2} P'_{u_0,v_0}(q), & \text{if } v^{-1}u \in W', \\ 0, & \text{otherwise}, \end{cases} \tag{9}$$

for all $(u,v) \in \mathrm{Int}(W)$, where l', u_0, and v_0 have the same meaning as in Proposition 2.3, and $P'_{u_0,v_0}(q)$ denotes the Kazhdan–Lusztig polynomial of u_0, v_0 as elements of W'. Note that it is possible that $u_0 \not\leqslant' v_0$ even if $u \leqslant v$. In this case $P'_{u_0,v_0}(q) = 0$ and so also $\mathcal{P}_{T'}(u,v) = 0$.

Theorem 5.3. *Let W' be a reflection subgroup of W and $\prec'$ be a reflection ordering of W'. Then $K_{T'}^{\prec'}$ is a W-kernel and $\mathcal{P}_{T'}$ is its KLS-function.*

Proof. It is clear that $\mathcal{P}_{T'}$ is unitary and that $\deg(\mathcal{P}_{T'}(u,v)) < l(u,v)/2$ if $u < v$ $(u,v) \in W)$. Therefore, by Theorem 2.1 and the definition of kernel, we only have to show that

$$K_{T'}^{\prec'} \mathcal{P}_{T'} = \overline{\mathcal{P}_{T'}} \quad \text{in } I(W). \tag{10}$$

Note that $(K_{T'}^{\prec'} \mathcal{P}_{T'})(u,v) = \mathcal{P}_{T'}(u,v) = 0$ if $v^{-1}u \notin W'$ (for if there is a directed path in $B_{T'}^{\prec'}(u,v)$ then necessarily $v^{-1}u \in W'$) so (10) certainly holds in this case. So suppose that $uW' = vW'$ and let w_0 be the element of minimal length in this coset. Then we have that

$$\left(K_{T'}^{\prec'} \mathcal{P}_{T'}\right)(u,v) = \sum_{u \leqslant a \leqslant v} K_{T'}^{\prec'}(u,a) \mathcal{P}_{T'}(a,v) = \sum_{a \in [u,v] \cap uW'} K_{T'}^{\prec'}(u,a) \mathcal{P}_{T'}(a,v).$$

Now, if $a \in [u,v] \cap uW'$ is such that $\mathcal{P}_{T'}(a,v) \neq 0$ and $K_{T'}^{\prec'}(u,a) \neq 0$ then $a_0 \leqslant' v_0$ and there is at least a directed path in $B_{T'}^{\prec'}(u,a)$. Therefore all the vertices of this path are in uW'. Hence, by Proposition 2.3, there is a corresponding path in $B(W')$ and so $u_0 \leqslant' a_0$. Therefore $u_0 \leqslant' a_0 \leqslant' v_0$. Conversely, if $b_0 \in W'$ is such that $u_0 \leqslant' b_0 \leqslant' v_0$ then, by Proposition 2.3, $u \leqslant w_0 b_0 \leqslant v$. Therefore

$$\begin{aligned}&\big(K_{T'}^{\prec'}\mathcal{P}_{T'}\big)(u,v)\\&\quad = q^{l(u,v)/2} \sum_{u_0 \leqslant' a_0 \leqslant' v_0} q^{-l'(a_0,v_0)/2} \sum_{\Gamma \in B_{T'}^{\prec'}(w_0u_0, w_0a_0)} \big(q^{1/2}-q^{-1/2}\big)^{l(\Gamma)} P'_{a_0,v_0}(q).\end{aligned}$$

But by Proposition 2.3 multiplication on the left by w_0 gives a bijection between $B_{T'}^{\prec'}(u,a)$ and $B_{T'}^{\prec'}(u_0,a_0)$ (where we consider u_0, a_0 as elements of W', and W' as a Coxeter system in its own right). Therefore, by Proposition 5.1 and Corollary 5.2,

$$\begin{aligned}&\big(K_{T'}^{\prec'}\mathcal{P}_{T'}\big)(u,v)\\&\quad = q^{l(u,v)/2} \sum_{u_0 \leqslant' a_0 \leqslant' v_0} q^{-l'(a_0,v_0)/2} \sum_{\Gamma \in B_{T'}^{\prec'}(u_0, a_0)} \big(q^{1/2}-q^{-1/2}\big)^{l(\Gamma)} P'_{a_0,v_0}(q)\\&\quad = q^{l(u,v)/2} \sum_{u_0 \leqslant' a_0 \leqslant' v_0} q^{-l'(u_0,v_0)/2} R'_{u_0,a_0}(q) P'_{a_0,v_0}(q)\\&\quad = q^{(l(u,v)-l'(u_0,v_0))/2} q^{l'(u_0,v_0)} P'_{u_0,v_0}\left(\frac{1}{q}\right)\\&\quad = q^{l(u,v)}\mathcal{P}_{T'}(u,v)\left(\frac{1}{q}\right)\end{aligned}$$

(where $R'_{u_0,a_0}(q)$ denotes the R-polynomial of u_0, a_0 as elements of W'), and the result follows. □

As a consequence of Theorem 5.3, we can now derive the results stated after Corollary 5.2. The first one can also be verified directly.

Corollary 5.4. *Let $t \in T$. Then K_t is a W-kernel and its KLS-function $\mathcal{P}_t$ is given by*

$$\mathcal{P}_t(u,v) = \begin{cases} q^{(l(u,v)-1)/2}, & \text{if } u \xrightarrow{t} v,\\ 1, & \text{if } u = v,\\ 0, & \text{otherwise,}\end{cases} \qquad \text{for all } (u,v) \in \mathrm{Int}(W). \tag{11}$$

Proof. Take $W' \stackrel{\mathrm{def}}{=} \{e,t\}$ in Theorem 5.3. □

Corollary 5.5. *Let W' be a dihedral reflection subgroup of W, and $\{s,t\}$ be its canonical generators. Then*:

(i) *$K_t K_{tst} \dots K_{sts} K_s$ is a W-kernel*;

(ii) *its KLS-function* $\mathcal{P}_{\langle s,t\rangle}$ *is given by*

$$\mathcal{P}_{\langle s,t\rangle}(u,v)=\begin{cases} q^{(l(u,v)-l'(u_0,v_0))/2}, & \text{if } v^{-1}u\in W' \text{ and } u_0\leqslant' v_0,\\ 0, & \text{otherwise,}\end{cases}$$

for all $(u,v)\in \mathrm{Int}(W)$;

(iii) $K_sK_{sts}\dots K_{tst}K_t=K_tK_{tst}\dots K_{sts}K_s$;

(iv) $\mathcal{P}_s\mathcal{P}_{sts}\dots\mathcal{P}_{tst}\mathcal{P}_t=\mathcal{P}_t\mathcal{P}_{tst}\dots\mathcal{P}_{sts}\mathcal{P}_s$.

In particular, the families $\{\mathcal{P}_t\}_{t\in T},\{K_t\}_{t\in T}\subseteq I(W)$ *satisfy the Yang–Baxter equations for* W.

Proof. (i) and (ii) follow directly from Theorem 5.3 using the reflection ordering $t\prec tst\prec \cdots \prec sts\prec s$ and the well-known fact (see, e.g., [15, Section 7.12]) that $P_{u,v}(q)=1$ for all $u\leqslant v$ in a dihedral Coxeter group. Regarding (iii), since the role of s and t is symmetrical we deduce from (i) that $K_sK_{sts}\dots K_{tst}K_t$ is also a W-kernel and from (ii) that $\mathcal{P}_{\langle s,t\rangle}$ is its KLS-function. But then $K_sK_{sts}\dots K_{tst}K_t$ and $K_tK_{tst}\dots K_{sts}K_s$ are two W-kernels that have the same KLS-function, hence they are equal.

To prove (iv), note that

$$(\mathcal{P}_t\mathcal{P}_{tst}\dots\mathcal{P}_{sts}\mathcal{P}_s)(u,v)=q^{l(u,v)/2}\sum_{\Gamma\in B^{\prec}_{T'}(u,v)}q^{-l(\Gamma)/2}.$$

Therefore,

$$(\mathcal{P}_s\mathcal{P}_{sts}\dots\mathcal{P}_{tst}\mathcal{P}_t)(u,v)=q^{l(u,v)/2}\sum_{\Gamma\in B^{\prec^*}_{T'}(u,v)}q^{-l(\Gamma)/2},$$

where $\prec^*$ is the ordering of $T'\stackrel{\mathrm{def}}{=}W\cap T$ which is the opposite of $\prec$. But by Proposition 5.1 (applied to $A=W'\cap T$) and part (iii) we have that

$$\sum_{\Gamma\in B^{\prec}_{T'}(u,v)}\left(q^{1/2}-q^{-1/2}\right)^{l(\Gamma)}=\sum_{\Gamma\in B^{\prec^*}_{T'}(u,v)}\left(q^{1/2}-q^{-1/2}\right)^{l(\Gamma)}$$

for all $(u,v)\in\mathrm{Int}(W)$. Since $\lim_{q\to 0^+}(q^{1/2}-q^{-1/2})=-\infty$ and $\lim_{q\to+\infty}(q^{1/2}-q^{-1/2})=+\infty$, this implies that

$$\sum_{\Gamma\in B^{\prec}_{T'}(u,v)}x^{l(\Gamma)}=\sum_{\Gamma\in B^{\prec^*}_{T'}(u,v)}x^{l(\Gamma)}\tag{12}$$

in $\mathbf{N}[x]$ for all $(u,v)\in\mathrm{Int}(W)$, and (iv) also follows. □

Corollary 5.5(iii) generalizes to any Coxeter group a special case of the main result of [7] (Theorem 3.1).

Note that the condition “$u_0 \leqslant' v_0$” cannot be omitted from Corollary 5.5(ii). For example, if $W = S_4$ and $W' = \{4231, 1324, 4321, 1234\}$ then $s = 1324$ and $t = 4231$ are the canonical generators of W' and $s \leqslant t$ in W. However, $s_0 = s \nleqslant' t = t_0$, so $\mathcal{P}_{\langle s,t\rangle}(s,t) = 0$.

We should mention that it is possible to give an explicit combinatorial formula for the polynomial in (12). In fact, by Proposition 2.3, the directed graph induced on W' by the Bruhat graph of W is isomorphic to the Bruhat graph of W' (as a Coxeter system). Using this one can show that

$$\sum_{\Gamma \in B^{\prec}_{T'}(u,v)} x^{l(\Gamma)} = \begin{cases} \displaystyle\sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} C_o(k, k-2i)x^{k-2i}, & \text{if } v^{-1}u \in W', \text{ and } u \leqslant' v, \\ 0, & \text{otherwise}, \end{cases} \tag{13}$$

where $k \stackrel{\text{def}}{=} l'(u_0, v_0)$ and $C_o(k, k-2i)$ denotes the number of compositions of k into $k-2i$ odd parts (so $C_o(k, k-2i) = \binom{k-i-1}{i}$ and $C_o(0,0) \stackrel{\text{def}}{=} 1$).

Note that it is not true that *any* product of the kernels K_t $(t \in T)$ is again a W-kernel. For example, if W is a noncommutative Coxeter group, and $s, t \in S$ are such that $st \neq ts$ then one can verify that

$$\bigl(K_s K_t \overline{K}_s \overline{K}_t\bigr)(e, st) = -q\bigl(q^{1/2} - q^{-1/2}\bigr)^2,$$

so $K_s K_t \overline{K}_s \overline{K}_t \neq \delta$, and hence, by Theorem 2.2, $K_s K_t$ is not a W-kernel. On the other hand, if $s, t \in S$ are such that $st = ts$ then it follows from Corollary 5.5 that $K_s K_t$ is a W-kernel. It would be interesting to know for which subsets $A \subseteq T$ and for which orderings $\prec$ of A the product $K^{\prec}_A$, defined by (8), is a W-kernel.

Since the Kazhdan–Lusztig kernel $\mathcal{R}$ is a product of the W-kernels $\{K_t\}_{t\in T}$ (Corollary 5.2), it is natural to wonder whether the KLS-function of $\mathcal{R}$ can be expressed in some way as a product in $I(W)$ of the elements $\{\mathcal{P}_t\}_{t\in T}$. Unfortunately, this is false. In fact, let Π be such a product. Then it is not hard to see that

$$\Pi(u,v) = q^{l(u,v)/2} \sum_{\Gamma} q^{-l(\Gamma)/2} \tag{14}$$

where Γ runs over some subset of the directed paths from u to v in $B(W)$. But if $[u, v]$ is a lattice (e.g., if $u = 2143$, $v = 4231$ in S_4) then it follows from [12, (the proof of) Proposition 3.3] that all the directed paths from u to v in $B(W)$ have length $l(u, v)$. Therefore, by (14), $\Pi(u, v)$ is a constant. On the other hand, by [3, Theorem 6.3] the Kazhdan–Lusztig polynomial of such a pair u, v equals the g-polynomial of the poset $[u, v]^*$ (see, e.g., [17, Section 3.14] for the definition of the g-polynomial of an Eulerian poset) and this is not always a constant (for example, $P_{2143,4231}(q) = 1 + q$).

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References

[1] A. Björner, F. Brenti, Combinatorics of Coxeter Groups, in: Grad. Texts Math., Springer-Verlag, 2003, to appear.
[2] T. Braden, R. MacPherson, From moment graphs to intersection cohomology, Math. Ann. 321 (2001) 533–551.
[3] F. Brenti, A combinatorial formula for Kazhdan–Lusztig polynomials, Invent. Math. 118 (1994) 371–394.
[4] F. Brenti, Kazhdan–Lusztig and R-polynomials from a combinatorial point of view, Discrete Math. 193 (1998) 93–116.
[5] F. Brenti, Lattice paths and Kazhdan–Lusztig polynomials, J. Amer. Math. Soc. 11 (1998) 229–259.
[6] F. Brenti, Twisted incidence algebras and Kazhdan–Lusztig–Stanley functions, Adv. Math. 148 (1999) 44–74.
[7] F. Brenti, S. Fomin, A. Postnikov, Mixed Bruhat operators and Yang–Baxter equations for Weyl groups, Int. Math. Research Notices 8 (1999) 419–441.
[8] I. Cherednik, Factorizing particles on a half line, and root systems, Teoret. Mat. Fiz. 61 (1984) 35–44.
[9] I. Cherednik, Quantum Knizhnik–Zamolodchikov equations and affine root systems, Comm. Math. Phys. 150 (1992) 109–136.
[10] J. Du, IC bases and quantum linear groups, in: Algebraic Groups and Their Generalizations: Quantum and Infinite-Dimensional Methods, in: Proc. Sympos. Pure Math., Vol. 56, Amer. Math. Society, Providence, RI, 1994, pp. 135–148.
[11] M. Dyer, Reflection subgroups of Coxeter systems, J. Algebra 135 (1990) 57–73.
[12] M.J. Dyer, On the "Bruhat graph" of a Coxeter system, Compositio Math. 78 (1991) 185–191.
[13] M. Dyer, Hecke algebras and shellings of Bruhat intervals, Compositio Math. 89 (1993) 91–115.
[14] R.M. Green, J. Losonczy, Canonical bases for Hecke algebra quotients, Math. Res. Lett. 6 (1999) 213–222.
[15] J.E. Humphreys, Reflection Groups and Coxeter Groups, in: Cambridge Stud. Adv. Math., Vol. 29, Cambridge Univ. Press, Cambridge, 1990.
[16] D. Kazhdan, G. Lusztig, Schubert varieties and Poincaré duality, in: Geometry of the Laplace Operator, in: Proc. Sympos. Pure Math., Vol. 34, Amer. Math. Society, Providence, HI, 1980, pp. 185–203.
[17] R.P. Stanley, Enumerative Combinatorics, Vol. 1, Wadsworth and Brooks/Cole, Monterey, CA, 1986.
[18] R.P. Stanley, Subdivisions and local h-vectors, J. Amer. Math. Soc. 5 (1992) 805–851.